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Rastall's and related theories are conservative gravitational theories although physically inequivalent to general relativity

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Abstract. We show the following: (1) the proper framework for testing Rastall's theory and its generalisations is in the case of non-negligible (i.e. discernible) gravitational effects such as gravity gradients; (2) these theories have conserved integral four-momentum and angular momentum; and (3) the Nordtvedt effect then provides limits on the parameters which arise as the result of the non-zero divergence of the energy–momentum tensor.

Let us reconsider those theories for which the energy–momentum tensor does not have zero divergence. The generic form of this type of theory was described by Rastall (1972) in that he showed that, to be compatible with the classical solar system experiments, the non-zero divergence must be proportional to the gradient of a scalar. The only constraint on the scalar is that it should go to a constant in asymptotically flat space. As an example, he proposed that the divergence of the energy–momentum tensor be

$$T^{\nu}_{\mu;\nu} = \lambda' R_{;\mu}, \tag{1}$$

where R is the scalar curvature and λ' is a constant parameter. The consistent field equations take the form

$$R_{\mu\nu} - (\lambda + \tfrac{1}{2}) g_{\mu\nu} R = -\kappa T_{\mu\nu}, \tag{2}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, κ is proportional to the gravitational constant, and $\lambda = \lambda'\kappa$. A complete post-Newtonian approximation of Rastall's theory is given by Smalley (1978a). Generalisations of Rastall's theories and other theories such as the Brans–Dicke theory (Brans and Dicke 1961) have been given by Smalley (1974a, b, 1975, 1976, 1977), Smalley and Prestage (1976) and Malin (1975).

Considerable discussion exists in the literature as to what the word 'conservative' means for the post-Newtonian approximation of metric gravitational theories. We generally subscribe to the definition of a metric gravitational theory given by Thorne *et al* (1973) of the form: there exists a space–time metric $g_{\alpha\beta}$; world lines are geodesics; and the Einstein equivalence principle is satisfied. The concept of a conservative gravitational theory is then equivalent to the question: is there a symmetric quantity $\theta_{\alpha\beta}$, called the stress–energy complex, which reduces to $T_{\alpha\beta}$ in flat space–time and whose ordinary divergence $\theta^{\alpha\beta}{}_{,\beta}=0$? If so, one can define in the usual manner the energy–momentum, four-vector P^α and the angular momentum tensor $J^{\alpha\beta}$ which are the desired *conserved* quantities. The answer to this question for Rastall's theory and several of its generalisations is positive (Smalley 1976, 1977, 1978a). However, these results vitiate the significance of the so-called non-conservative parameters ζ_1, ζ_2, ζ_3, ζ_4 (Nordtvedt and Will 1972, Will and Nordtvedt 1972), some of which are non-zero in the above theories.

Lindblom and Hiscock (1982) have correctly shown that if we assume that the perfect fluid energy–momentum tensor (with energy density ε, pressure p, and four-velocity U^μ)

$$t^{\mu\nu}=(\varepsilon+p)U^\mu U^\nu+pg^{\mu\nu}, \tag{3}$$

is proportional to the Einstein tensor, $G^{\mu\nu}$, then it is possible to rescale the field equation (2), so that it is equivalent to Einstein's gravitational theory with a perfect fluid, but now the *measured* energy density and pressure are given by

$$\varepsilon'=(1+4\lambda)^{-1}[(1+3\lambda)\varepsilon+3\lambda p], \tag{4}$$

$$p'=(1+4\lambda)^{-1}[(1+\lambda)p+\lambda\varepsilon]. \tag{5}$$

(Note that in our convention there is a negative sign on the right-hand side of (2) which means that we have a sign difference for λ in (4) and (5) compared with Lindblom and Hiscock. Also they do not distinguish between λ and λ' in (1) and (2).) This is, of course, a possible interpretation but does not lead to anything new. This parallels the analysis of Harrison (1972) who has shown that it is always possible, using conformal transformations, to relate physically inequivalent theories. Alternatively, Lindblom and Hiscock considered the identification of the $T_{\mu\nu}$ of (2) with the $t_{\mu\nu}$ given by (3) and sought physical limits on the λ parameter from the behaviour in the laboratory of a near perfect fluid such as helium gas at low temperature. Experimental limits from solar system experiments have shown that necessarily $\lambda\leqslant O(\nu^2)$ (Smalley 1978a). Upon using (1) and (2), the Bianchi identities yield the following equations of motion for the physical variables ε and p:

$$U^\mu\nabla_\mu[(1+3\lambda)\varepsilon+3\lambda p]+(1+4\lambda)(\varepsilon+p)\nabla_\mu U^\mu=0, \tag{6}$$

$$(1+4\lambda)(\varepsilon+p)U^\mu\nabla_\mu U^\nu+(g^{\mu\nu}+U^\mu U^\nu)\nabla_\mu[(1+\lambda)p+\lambda\varepsilon]=0. \tag{7}$$

Lindblom and Hiscock attempt to analyse these equations in what they refer to as the Newtonian limit ($U^\mu=(1,v^i)$, $v^i\ll 1$, $p\ll\varepsilon$, and negligible self-gravitational effects). Eventually they find a wave-like equation for small density fluctuations $\delta\varepsilon$,

$$\partial_0\partial_0(\delta\varepsilon)-v_s^2\partial^i\partial_i(\delta\varepsilon)=0, \tag{8}$$

with propagation velocity

$$v_s^2=\partial p/\partial\varepsilon+\lambda c^2, \tag{9}$$

providing $\lambda\ll 1$.

By comparison with gaseous helium near 4 K, they conclude that $\lambda\leqslant 0.01\ v_s^2/c^2\approx 10^{-15}$ compared with $\lambda\approx 10^{-9}$ obtained by Smalley (1978a) from solar system constraints or velocities. They also claim that their results would apply to more general theories too. Thus, upon face value, the Rastall-type theories appear to be non-viable alternatives to general relativity.

In what follows, we show explicitly the analysis which leads to (8)–(9) must either neglect *all* gravitational effects or it must retain them to appropriate orders which, in either case, does not lead to their results. Furthermore, an example of a more general theory yields the classical sound velocity. Finally we show by example that the Rastall theories will provide theoretical limits in the interpretation of solar system experiments, for example, lunar laser ranging.

In order to carry out a consistent calculation for the deviations of ε and p from their equilibrium values beginning with the fluid equations (6)–(7), we must establish an approximation scheme in powers of ν for the various quantities. In this case we have (for $c = 1$)

$$\lambda \sim \mathrm{O}(\nu^2), \tag{10}$$

$$\varepsilon = \varepsilon^{(2)} + \varepsilon^{(4)} + \ldots, \tag{11}$$

$$p = p^{(4)} + p^{(6)} + \ldots = p^{(4)} + (\partial p/\partial \varepsilon)\varepsilon^4 + \ldots, \tag{12}$$

where the superscript in parentheses refers to order of ν and $\partial p/\partial\varepsilon \sim \mathrm{O}(\nu^2)$. The equilibrium values, $\varepsilon^{(2)}$ and $p^{(4)}$, are assumed uniform. This has the effect of removing *some* of the external gravitational field effects that would enter through the $g^{\mu\nu}$ term in (7). In (12), the pressure has been expanded as a function of ε, and the partial derivative is understood to be taken at constant entropy. To the correct order, we also have $\partial_k p^{(6)} = (\partial p/\partial\varepsilon)\partial_k \varepsilon^{(4)}$. Finally, we identify $\varepsilon^{(4)}$ with the small density fluctuation $\delta\varepsilon$ of (8).

After taking the divergence of (7) and subtracting the covariant derivative of (6) along the four-velocity U, we obtain the equation

$$\begin{aligned} 2\partial U^\mu \nabla_\nu[(1+\lambda)p + \lambda\varepsilon]\nabla_\mu U^\nu &+ (1+4\lambda)(\varepsilon + p)(\nabla_\nu U^\mu)\nabla_\mu U^\nu + (1+4\lambda)(\varepsilon+p)U^\mu U^\nu R_{\mu\nu} \\ &+ g^{\mu\nu}\nabla_\nu\nabla_\mu[(1+\lambda)p + \lambda\varepsilon] - (\nabla_\nu U^\nu)U^\mu\nabla_\mu[3\lambda p + (1+3\lambda)\varepsilon] \\ &+ U^\mu U^\nu \nabla_\nu\nabla_\mu[(1-2\lambda)p - (1+2\lambda)\varepsilon] = 0. \end{aligned} \tag{13}$$

It is crucial to note that we have used the important commutator identity for covariant derivatives

$$2U^\mu \nabla_{[\nu}\nabla_{\mu]}U^\nu = R_{\mu\nu}U^\mu U^\nu, \tag{14}$$

in the calculation of (13). For the specific example of low-temperature helium gas used by Lindblom and Hiscock, it is reasonable to neglect streamline velocities which are of the order of $1\ \mathrm{m\,s^{-1}}$ in comparison with sound velocities which are near $100\ \mathrm{m\,s^{-1}}$. Separating out the $\mathrm{O}(v^6)$ part of (13) (after using lower-order identities from (6), (7) and (13)), we finally obtain

$$\partial_0\partial_0\varepsilon^{(4)} - (\partial p/\partial\varepsilon + \lambda)\nabla^2\varepsilon^{(4)} - (\varepsilon^{(4)} + p^{(4)} + 4\lambda\varepsilon^{(2)})R^{(2)}_{00} - \varepsilon^{(2)}R^{(4)}_{00} = 0, \tag{15}$$

which unlike (8) is a wave equation *with sources.* It is tempting to argue (as Lindblom and Hiscock do) that the time components of the Ricci tensor can be neglected for systems with negligible self gravity. If so, then we have negated the premise of the theory upon which are based the *gravitational* field equations (2) along with the divergence condition on the energy–momentum tensor. That is, it is the field equations that allow you to form a divergentless $T^{\mu\nu}$, constructed from ε' and p' given by (4) and (5), which also lead to the fluid equations (6) and (7). Indeed, the field equations (to the proper order) yield

$$R_{00}^{(2)} = -4\pi\varepsilon^{(2)}, \tag{16}$$

$$R_{00}^{(4)} = -4\pi(\varepsilon^{(4)} + 3p^{(4)} - \varepsilon^{(2)} g_{00}^{(2)}), \tag{17}$$

so that (15) finally becomes

$$\partial_0\partial_0\varepsilon^{(4)} - (\partial p/\partial\varepsilon + \lambda)\nabla^2\varepsilon^{(4)} + 4\pi\varepsilon^{(2)}(2\varepsilon^{(4)} + 4p^{(4)} + 4\lambda\varepsilon^{(2)} - \varepsilon^{(2)} g_{00}^{(2)}) = 0. \tag{18}$$

It is interesting to note that even though one assumes there is negligible self gravity, it is not possible to eliminate the curvature terms (because of the assumed structure of the field equation); and further, the external field of the Earth also enters through $g_{00}^{(2)} = -2U(x)$ in the source term. Alternatively, to set the Ricci terms in (15) to zero requires for consistency that the scalar curvature term of (1) in the matter equation of motion also vanish. But in this case, λ does not occur at all in the matter equations. This is the consistent flat space (non-gravitational) limit of Rastall's theory (1972) in which he requires that the divergence of the energy–momentum tensor be proportional to a vector which vanishes in asymptotic flat-space.

Consider now a generalisation of Rastall's theory, in particular, the case of a modified Brans–Dicke theory (Smalley 1975, 1976, 1977) in which

$$T_{\mu}{}^{\nu}{}_{;\nu} = (\sigma/8\pi)\phi_{,\mu}\phi^{,\nu}\phi_{,\nu}, \tag{19}$$

where σ is a parameter of O(1). The scalar field ϕ is governed by the field equation

$$\Box^2\phi = E^{-1}(\phi)T, \tag{20}$$

where $E(\phi)$ is the function

$$E(\phi) = -\sigma\phi^2/4\pi + c, \tag{21}$$

obtained from consistency with the Bianchi identities. The field equations are

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R = -(8\pi/\phi)(T^{\mu\nu} + T_{\phi}^{\mu\nu}), \tag{22}$$

where

$$T_{\phi}^{\mu\nu} = A(\phi)\phi^{,\mu}\phi_{,\nu} + B(\phi)g^{\mu\nu}\phi^{,\rho}\phi_{,\rho} + C(\phi)\phi^{,\mu;\nu} + D(\phi)g^{\mu\nu}\Box^2\phi, \tag{23}$$

with

$$A(\phi) = -2B(\phi) = \frac{1}{8\pi\phi}\left[\omega + \frac{\sigma}{G^2}\left(\frac{2\omega+4}{2\omega+3}\right)^2 - \sigma\phi^2\right], \tag{24}$$

$$C(\phi) = -D(\phi) = 1/8\pi, \tag{25}$$

where ω is the usual Brans–Dicke parameter. Gauge conditions have been used to set the constant C through the relation

$$8\pi C - 2\sigma\mathscr{S}^2 \equiv 2\omega + 3, \tag{26}$$

and the renormalised gravitational constant becomes (because of the Newtonian limit)

$$G = \mathscr{S}[(2\omega+4)/(2\omega+3)] \equiv 1. \tag{27}$$

Note that the identification

$$\hat{\omega} = \omega + \sigma[(2\omega+4)/(2\omega+3)]^2 - \sigma\phi^2 \tag{28}$$

formally reduces the structure of the field equations to the form of the usual Brans–Dicke theory, but now $\hat{\omega}=\hat{\omega}(\phi)$. The theory looks formally equivalent to the Bergmann–Wagoner theory (Bergmann 1968, Wagoner 1970) and the Nordtvedt (1970) scalar–tensor theory, except that the cosmological constant $\Lambda=0$ and the $\mathrm{d}\hat{\omega}/\mathrm{d}\phi$ term is missing in the $\Box^2\phi$ equation as a result of $T^{\mu\nu}{}_{,\nu}\neq 0$. The important point here is that this did not involve a redefinition of $T_{\mu\nu}$ (as in Rastall's theory). In fact, the analysis for the sound velocity equation yields

$$\partial_0\partial_0\varepsilon^{(4)}-\frac{\partial p}{\partial\varepsilon}\nabla^2\varepsilon^{(4)}-(\varepsilon^{(4)}+p^{(4)})R^{(2)}_{00}-\varepsilon^{(2)}R^{(4)}_{00}+\frac{\sigma}{8\pi}\partial_j(\phi^{,j}\phi_{,k}\phi^{,k})=0. \tag{29}$$

Since $\phi_{,j}\propto\partial_j U$ (the gradient of the gravitational field), it is not difficult to see that for both negligible internal and external gravitational effects the usual classical result is obtained without any ambiguities as in the pure Rastall case.

Finally, we show for the example of a Rastall-type theory given by (19)–(27) that the unambiguous experiment is in the astrophysical setting. The parametrised post-Newtonian (PPN) parameters for this theory (Smalley 1975) are

$$\gamma=\frac{\omega+1}{\omega+2},\qquad \beta=1,\qquad \zeta_w=0,\qquad \Delta_1=\frac{7\omega+10}{7\omega+14},\qquad \Delta_2=1,$$

$$\beta_1=\frac{2\omega+3}{2\omega+4},\qquad \beta_2=\frac{2\omega+1}{2\omega+4}+\frac{4\sigma}{(2\omega+3)^3},\qquad \beta_3=1,\qquad \beta_4=\frac{\omega+1}{\omega+2}. \tag{30}$$

Only the β_2 parameter is altered. This is also equivalent to the non-zero ζ parameter

$$\zeta_2=8\sigma/(2\omega+3)^3; \tag{31}$$

yet this theory has integral conservation laws for four-momentum and angular momentum (Smalley 1976). Recent experimental limits on the quadrupole moment of the Sun (see e.g. Gough 1982) have severly restricted the Brans–Dicke theory by constraining the parameter $\omega\gtrsim 100$. There have been attempts to use also the Nordtvedt effect (Nordtvedt 1968a, b, 1973) to impose experimental limits on ω. The Nordtvedt effect is represented by the parameter

$$\begin{aligned}\eta&=4\beta+3\gamma-7\Delta_1-\tfrac{1}{3}(2\beta+2\beta_2-3\gamma+\Delta_2-2)-\tfrac{13}{3}\zeta_w\\&=4\beta-\gamma-3-\alpha_1+\tfrac{2}{3}\alpha_2-\tfrac{2}{3}\zeta_1-\tfrac{1}{3}\zeta_2-\tfrac{13}{3}\zeta_w,\end{aligned} \tag{32}$$

where in the second line we have noted the Will–Nordtvedt PPN parameters. Experimentally $\eta=0.00\pm0.03$ (Williams *et al* 1976, Shapiro *et al* 1976).

However, in this modified Brans–Dicke theory, we obtain instead

$$\eta=(\omega+2)^{-1}-\tfrac{8}{3}\sigma/(2\omega+3)^3. \tag{33}$$

We find, therefore, that (31) does not imply just a constraint on ω, but an experimental limit on the parameter σ. On face value, if we take $\eta=0$,

$$\sigma=\tfrac{3}{8}(2\omega+3)^3/(\omega+2). \tag{34}$$

But from (30) we discover that this is precisely the value of σ necessary to force $\beta_2=1$; i.e. β_2 takes its general relativity value.

Other examples are known (Smalley 1977), for example, the Rastall-type theory in which

$$T_\mu{}^\nu{}_{;\nu}=\delta(UR)_{;\mu}. \tag{35}$$

One obtains the usual PPN parameters of general relativity except that here

$$\zeta_2 = -2\delta. \tag{36}$$

As a result

$$\eta = 4\beta - \gamma - 3 + \tfrac{2}{3}\delta, \tag{37}$$

so that the Nordtvedt parameter η given above is not just a limit on the Robertson parameters, β and γ, but must be considered a limit on the three parameters β, γ and δ.

In summary, we mention that it is generally held that $T^{\mu\nu}{}_{;\nu} = 0$ satisfies special relativity locally independent of the PPN metric because it is always possible to choose a coordinate system *at a point* where the connection vanishes and the metric is locally Lorentzian. This is of course true *but only up to* curvature or gravity-gradient terms. This is the gist of our first considerations. There is another way to put this. The Einstein equivalence principle demands that in local Lorentz frames, the laws of special relativity follow. Indeed, we must interpret this as an exact statement only at a point and as only an approximate statement over an extended (albeit small) region about the point when gravity is present. (An exception is the special case of uniform gravitational fields which are global so that local Lorentz frames can be extended to the global Minkowski flat space–time.) Abruptly we must now change our viewpoint to consider the exact nature of the gravitational fields over the volume of the experiment itself, i.e. we must test for flatness (Misner *et al* 1973). It is at this point that these non-zero divergence theories depart from the usual formulations of relativity. That is, we do not require that $T^{\mu\nu}{}_{,\nu} = 0$ in the extended local Lorentz frame. We look instead for conserved four-momentum angular momentum in the global post-Newtonian frame, or equivalently for a quantity $\theta^{\alpha\beta}$, called the stress–energy complex, which reduces to $T^{\alpha\beta}$ in flat space–time and whose ordinary divergence $\theta^{\alpha\beta}{}_{,\beta} = 0$ (Smalley 1976, 1977, 1978a). Even if $T^{\mu\nu}{}_{;\nu} = 0$, it is not possible to obtain integral conservation laws directly because of the presence of the Christoffel symbols in the covariant derivative (Landau and Lifshitz 1962). In fact, we now know that the zero divergence of the energy–momentum tensor is not even a necessary condition. In all the cases discussed above, conserved quantities were found with the proper limit to the local Lorentz frame. At some point in that local frame, the complex reduces to $T^{\mu\nu}$ and $T^{\mu\nu}{}_{,\nu} = 0$, but over the extended local frame, the gravitational fields exert their influence (curvature, gravity gradients) unless they can be neglected for a particular experiment. The Einstein equivalence principle is certainly compatible with this, but now one *cannot* make the converse requirement that the Einstein equivalence principle implies that $T^{\mu\nu}{}_{,\nu} = 0$ implies that $T^{\mu\nu}{}_{;\nu} = 0$. This is certainly known not to be the case for theories with torsion (Smalley 1978b). Thus, mathematically these theories are consistent. But how well do they compare with the motion of matter? When gravity is ignorable, we obtain the usual results as shown above. It is also easy to show that for the cases where gravity is important, then for all three cases, the continuity equation holds. The usual Euler equation is obtained for the second example (Smalley 1975), but there is a density-gradient term proportional to λ for the first example (Smalley 1978a) and proportional to δ for the third example (Smalley 1977). It is difficult to see how these theories could be tested in a local experiment in which the gravitational field is negligible. The effect in solar experiments may be debatable, but they could lead to important consequences in the treatment of relativistic stars.

Finally, we mention that the equation of an action principle for these theories is under investigation. Generally speaking, we expect that the laws of physics should be derivable from an action principle. Our understanding of these theories will be greatly enhanced if an action principle can be found.

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References

Bergman P G 1968 *Int. J. Theor. Phys.* **1** 25
Brans C H and Dicke R H 1961 *Phys. Rev.* **124** 925
Gough D O 1982 *Nature* **298** 334
Harrison E R 1972 *Phys. Rev.* D **6** 2077
Landau L D and Lifshitz E M 1962 *Classical Theory of Fields* (Reading, MA: Addison-Wesley) 2nd edn
Lindblom L and Hiscock W A 1982 *J. Phys. A: Math. Gen.* **15** 1827
Malin S 1975 *Phys. Rev.* D **11** 707
Misner C M, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman) p 30
Nordtvedt K Jr 1968a *Phys. Rev.* **169** 1014
—— 1968b *Phys. Rev.* **170** 1186
—— 1970 *Astrophys. J.* **161** 1059
—— 1973 *Phys. Rev.* D **7** 2347
Nordtvedt K Jr and Will C M 1972 *Astrophys. J.* **177** 775
Rastall P 1972 *Phys. Rev.* D **6** 3357
Shapiro I I, Counselman C C III and King R W 1976 *Phys. Rev. Lett.* **36** 555
Smalley L L 1974a *Phys. Rev.* D **9** 1635
—— 1974b *Lett. Nuovo Cimento* **10** 231
—— 1975 *Phys. Rev.* D **12** 376
—— 1976 *Phys. Lett.* **57A** 300
—— 1977 *Scientific Application of Lunar Laser Ranging* (Dordrecht, Holland: Reidel) pp 91–102
—— 1978a *Found. Phys.* **8** 59
—— 1978b *Phys. Rev.* D **18** 3896
Smalley L L and Prestage J 1976 *Nuovo Cimento* B **35** 224
Thorne K S, Lee D L and Lightman A P 1973 *Phys. Rev.* D **7** 3563
Wagoner R V 1970 *Phys. Rev.* D **1** 3209
Will C M and Nordtvedt K Jr 1972 *Astrophys. J.* **177** 757
Williams J G, Dicke R H, Bender P L, Alley C O, Carter W E, Currie D G, Eckhardt D H, Faller J E, Kaula W M, Mulholland J D, Plotkin H H, Poultney S K, Shelus P J, Silberberg E D, Sinclair W S, Slade M A and Wilkinson D T 1976 *Phys. Rev. Lett.* **36** 551